

MATH 138 - Calculus II

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1 Review

A few useful identities involving summation notation

1.

$$\sum_{i=1}^n i = 1 + 2 + 3 + \cdots + n = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n/2 \text{ times}} = \frac{n(n+1)}{2}$$

2.

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

3.

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

We call i the *index of summation*.

4. Let $f(x) = 2 + 3x^2$, then

$$\sum_{i=2}^4 f(i) = \sum_{i=2}^4 (2 + 3i^2) = 2 + 3(2)^2 + 2 + 3(3)^2 + 2 + 3(4)^2 = 93$$

5. *Geometric series:*

$$\sum_{i=0}^{n-1} ar^i = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

6. *Reindexing:*

$$\sum_{i=1}^5 \frac{(-1)^{i+1}}{i^2} = \sum_{i=0}^4 \frac{(-1)^{i+2}}{(i+1)^2}$$

7. *Theorem:*

$$\sum_{i=1}^n (cf(x_i) + dg(x_i)) = c \sum_{i=1}^n f(x_i) + d \sum_{i=1}^n g(x_i)$$

for all constants c, d .

1.1 Estimating the area of regions bounded by a curve $y = f(x)$ and the x -axis

Let $f(x) = x^2$ on the interval $[1, 10]$. For the area under the curve, we subdivide the interval into n small intervals of size Δx and make rectangles of heights $f(x_i)$. Then an approximate area under the curve is given by the sum of the area of these approximating rectangles.

That is, $\Delta x = (10 - 1)/n = 9/n$ and the *sample points* are $x_i = 1 + i\Delta x = 1 + 9i/n$, so

$$\begin{aligned}
 \text{Approximate area} &= \sum_{i=1}^n A_i = \sum_{i=1}^n f(x_i) \cdot \Delta x \\
 &= \sum_{i=1}^n (1 + 9i/n)^2 \cdot 9/n \\
 &= \frac{9}{n} \sum_{i=1}^n \left(1 + \frac{18i}{n} + \frac{81i^2}{n^2} \right) \\
 &= \frac{9}{n} \left(n + \frac{18}{n} \sum_{i=1}^n i + \frac{81}{n^2} \sum_{i=1}^n i^2 \right) \\
 &= \frac{9}{n} \left(n + \frac{18}{n} \frac{n(n+1)}{2} + \frac{81}{n^2} \frac{n(n+1)(2n+1)}{6} \right) \\
 &= \frac{9}{n} \left(n + 9(n+1) + \frac{27(n+1)(2n+1)}{2n} \right) \\
 &= 9 + 81 + \frac{81}{n} + \frac{243(2n^2 + 3n + 1)}{2n^2} \\
 &= 90 + \frac{81}{n} + 243 + \frac{243(3)}{2n} + \frac{243}{2n^2}
 \end{aligned}$$

The actual area is given by allowing the number of rectangles to become infinite (hence, arbitrarily small), thus letting n tend to infinity. That is, the actual area is found as

$$\lim_{n \rightarrow \infty} \left(90 + \frac{81}{n} + 243 + \frac{243(3)}{2n} + \frac{243}{2n^2} \right) = 90 + 243 = 333.$$

Note: the choice of sample points were the right hand sides of the rectangles, which gave us an over-estimate. One could also have chosen the left-hand sides of the rectangles and obtained an under-estimate and both would have the same limit.

2 The Definite Integral

Let $f(x)$ be a continuous function on $[a, b]$ and let $\Delta x = \frac{b-a}{n}$ for a known integer $n > 0$. Let x_i^* be a sample point in each Δx interval, then the expression $\sum_{i=1}^n f(x_i^*) \Delta x$ is called a *Riemann sum* over $[a, b]$. The number obtained by $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ is called a *definite integral* of $f(x)$ over $[a, b]$. The definite integral is then denoted by

$$\boxed{\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x} \tag{1}$$

where

- $f(x)$ is the *integrand*
- $\int$ is the *integral sign*
- a, b are the *limits of integration*

and this process is called integration. If $f(x)$ is a function on $[a, b]$ such that $\int_a^b f(x)dx$ exists, we say that $f(x)$ is *integrable* on $[a, b]$.

Theorem 1. *If $f(x)$ is continuous on $[a, b]$, then it is integrable on $[a, b]$ (i.e. $\int_a^b f(x)dx$ exists).*

Definition 1. *If $f(x)$ is continuous and non-negative on $[a, b]$, then we define the area of the region bounded by $f(x)$ and the x -axis over $[a, b]$ as being the positive number obtained by $\int_a^b f(x)dx$. In the case that $f(x)$ is negative, we take the absolute value and if $f(x) \geq 0$ on $[a, c]$ then $f(x) \leq 0$ on $[c, b]$ where $a < c < b$, then the area is $\int_a^c f(x)dx + \left(-\int_c^b f(x)dx\right)$.*

Example 1. *Using Riemann sums to find a formula for $\int_a^b xdx$ (where $0 < a < b$).*

$$\begin{aligned}
\int_a^b xdx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(a + \frac{(b-a)i}{n} \right) \cdot \frac{b-a}{n} \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n (a\Delta x + i\Delta x^2) \\
&= \lim_{n \rightarrow \infty} \left(a\Delta x \sum_{i=1}^n 1 + \Delta x^2 \sum_{i=1}^n i \right) \\
&= \lim_{n \rightarrow \infty} \left(a\Delta x n + \Delta x^2 \frac{n(n+1)}{2} \right) \\
&= \lim_{n \rightarrow \infty} \left(a(b-a) + \frac{(b-a)^2}{n^2} \cdot \frac{n^2+n}{2} \right) \\
&= a(b-a) + \lim_{n \rightarrow \infty} \left(\frac{(b-a)^2}{2} + \frac{(b-a)^2}{2n} \right) \\
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= \frac{b^2 - a^2}{2}
\end{aligned}$$

2.1 Properties of the definite integral

We provide the following list of properties without proof. Note that, geometrically, these results should be intuitive.

1.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

3.

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

4.

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$$

5. if $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)dx \geq 0$$

6. if $f(x)$ is *odd* (i.e. $f(-x) = -f(x)$), then

$$\int_{-a}^a f(x)dx = 0$$

7. if $f(x)$ is *even* (i.e. $f(-x) = f(x)$), then

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$$

8. If $f(x) \geq g(x)$ on $[a, b]$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx,$$

9. If $m \leq f(x) \leq M$ for all x on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

10.

$$\int_a^a f(x)dx = 0$$

Example 2. Show that $\int_2^3 \frac{1}{x^2+1}dx$ is in the interval $[1/10, 1/5]$.

Note, for $x \in [2, 3]$ that $x^2 + 1 \in [5, 10]$ or equivalently that $5 \leq x^2 + 1 \leq 10$ for all $x \in [2, 3]$. Hence, also,

$$m := 1/10 \leq \frac{1}{x^2+1} \leq 1/5 =: M$$

and our properties guarantee, that

$$\frac{1}{10} = \frac{3-2}{10} \leq \int_2^3 \frac{1}{x^2+1}dx \leq \frac{1}{5}.$$

Example 3. Evaluate; $\int_{-1}^3 2|x|dx = 2 \left[\int_{-1}^0 -x dx + \int_0^3 x dx \right] = 2(\frac{1}{2} + \frac{9}{2})$

2.2 Average Value of a Function

Suppose $f(x)$ is integrable on $[a, b]$ with $a < b$. The *average value* of f on $[a, b]$ is defined as

$$\boxed{A(f)_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)dx} \quad (2)$$

Example 4. Find the average value of $f(x) = 100 - 3x^2$ on $[1, 5]$.

$$A(f)_{[1,5]} = \frac{1}{5-1} \int_1^5 (100 - 3x^2)dx = \frac{1}{4}(276) = 69$$

Theorem 2 (Average-value Theorem). *Suppose f is continuous on $[a, b]$, with $a < b$. Then there exists an $x^* \in [a, b]$ such that $f(x^*) = A(f)_{[a, b]}$. That is, a continuous function actually achieves its averages value somewhere within the interval $[a, b]$.*

Proof. Since $f(x)$ is continuous there exist a minimum and maximum at some $c, d \in [a, b]$ respectively. Then, certainly $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$. Now, from property 9, we have

$$f(c)(b-a) \leq \int_a^b f(x)dx \leq f(d)(b-a)$$

or equivalently (dividing by $(b-a)$)

$$f(c) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f(d).$$

Now, by the intermediate value theorem, there exists $x^* \in [c, d]$ such that $f(x^*) = \frac{1}{b-a} \int_a^b f(x)dx$. □

2.3 Fundamental Theorem of Calculus - Part I

Theorem 3 (FTC I). *Suppose $f(x)$ is continuous on $[a, b]$. Let $F(x)$ be a function on $[a, b]$ defined as $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$.*

Proof. Let $x \in [a, b]$ and proceed computing the derivative by first principles.

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t)dt \end{aligned}$$

By the average value theorem there is an $x^* \in [x, x+h]$ so that $f(x^*) = \frac{1}{h} \int_x^{x+h} f(t)dt$ so then

$$F'(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t)dt = \lim_{h \rightarrow 0} f(x^*) = \lim_{x^* \rightarrow x} f(x^*) = f\left(\lim_{x^* \rightarrow x} x^*\right) = f(x)$$

as required. □

Example 5. 1.

$$\frac{d}{dt} \int_0^t \sqrt{x^2 + 1} dt = \sqrt{t^2 + 1}$$

2.

$$\frac{d}{dt} \int_t^{-3} \sin^2(x) dx = -\frac{d}{dt} \int_{-3}^t \sin^2(x) dx = -\sin^2(t)$$

3.

$$\begin{aligned} \frac{d}{dt} \int_0^{t^2} \cos(x^2) dx &= \frac{d}{dt} \int_0^{u(t)} \cos(x^2) dx \\ &= \frac{d}{du} \int_0^u \cos(x^2) dx \cdot \frac{du}{dt} \\ &= \cos(u^2) \cdot 2t \\ &= 2t \cos(t^4) \end{aligned}$$

4.

$$\frac{d}{dt} \int_{-2t}^t \frac{1}{1+x^2} dx = \frac{d}{dt} \left(- \int_0^{-2t} \frac{1}{1+x^2} dx + \int_0^t \frac{1}{1+x^2} dx \right) = -\frac{-2}{1+4t^2} + \frac{1}{1+t^2}$$

Lemma 1. If f and g are two functions whose derivative is F , then $f(x) = g(x) + c$

Proof. Given that $f'(x) = F(x) = g'(x)$, then

$$\frac{d}{dx}(f(x) - g(x)) = F(x) - F(x) = 0 \Rightarrow f(x) - g(x) = c$$

since the derivative of a constant is 0. □

2.4 Fundamental Theorem of Calculus II

Definition 2. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Theorem 4 (FTC II). Suppose $f(x)$ is continuous on $[a, b]$ and let $F(x)$ be any anti-derivative of $f(x)$ (i.e. $F'(x) = f(x)$). Then

$$\boxed{\int_a^b f(x) dx = F(b) - F(a)} \quad (3)$$

Proof. Given that $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$, let $G(x) = \int_a^x f(t) dt$ so that $G'(x) = f(x)$ too. By the previous lemma, then, there is some constant c such that $G(x) = F(x) + c$. Now,

$$\int_a^b f(t) dt = G(b) = G(b) - 0 = G(b) - G(a) = (F(b) + c) - (F(a) + c) = F(b) - F(a)$$

□

This theorem says that knowing any anti-derivative $f(x)$ allows us to compute any definite integral.

Example 6. Now we are able to compute a few more definite integrals like

$$\int_2^3 e^x dx = e^x \Big|_2^3 = e^3 - e^2$$

and

$$\int_{-7}^{500} \sin x dx = -\cos(500) - (-\cos(-7))$$

Notation: If $F(x)$ is an antiderivative of $f(x)$ denote by $\int f(x) dx$ the family of antiderivatives $F(x) + C$ and call this the *indefinite integral* of $f(x)$.

2.5 A few antiderivatives

- $\int k dx = kx + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$

- $\int \sec^2 x dx = \tan x + C$
- $\int \sec x \tan x dx = \sec x + C$

Example 7. Find the region bounded by $\cos x$ and the x -axis over the interval $[0, \pi]$.

Note: $\int_0^\pi \cos x dx$ won't work since the function is negative on the second half of this interval. Must use,

$$A = A_1 + A_2 = \int_0^{\pi/2} \cos x dx + \left(- \int_{\pi/2}^\pi \cos x dx \right) = \sin x|_0^{\pi/2} - \sin x|_{\pi/2}^\pi = (1 - 0) - (0 - 1) = 2$$

Example 8. $\int_{-1}^1 \frac{1}{x^2} dx$ does not exist since $\frac{1}{x^2}$ is not continuous at 0 and (3) does not apply.

Definition 3. Let $y = f(x)$ be continuous, we define the differential of y , denoted by dy , as being $dy = f'(x)\Delta x$ where $\Delta x = x_1 - x$ and $x_1 > x$.

Observation: if $y = x$, then $dy = dx = \frac{dx}{dx} \cdot \Delta x = \Delta x$ so $\Delta x = dx$ and so, if $y = f(x)$ then $dy = f'(x)dx$.

3 Techniques of integration

3.1 Integration by Substitution

Theorem 5 (The substitution rule). Let $u = g(x)$ be differentiable and having as its image, the interval I . Suppose $f(x)$ is continuous on the interval I . Then,

$$\boxed{\int f(g(x)) \cdot g'(x) dx = \int f(u) du} \quad (4)$$

Proof. Given that $u = g(x)$, we are required to show that the general antiderivative of $f(g(x)) \cdot g'(x)$ is the same as the general antiderivative (AD) of $f(u)$.

Suppose $F(u)$ is an AD of $f(u)$, then $\int f(u) du = F(u) + C = F(g(x)) + C$.

Now we claim that $\int f(g(x))g'(x) = F(g(x)) + C$. Indeed, it remains to show that $\frac{d}{dx}(F(g(x)) + C) = f(g(x))g'(x)$ which is true by the chain rule so the theorem holds. $\square$

Remark 1. What the substitution rule is saying is that

$$\int \underbrace{f(g(x))}_{f(u)} \underbrace{g'(x) dx}_{du}$$

and that's it!

Example 9. letting $u = x^3 + 9$ so that $du = 3x^2 dx$ which is equivalent to $x^2 dx = \frac{1}{3} du$, we find

$$\begin{aligned} \int_0^3 x^2 \sqrt{x^3 + 9} dx &= \int_{x=0}^{x=3} \frac{1}{3} \sqrt{u} du \\ &= \frac{1}{3} \left(\frac{2}{3} u^{3/2} \right) \Big|_{x=0}^{x=3} \\ &= \frac{2}{9} (x^3 + 9)^{3/2} \Big|_0^3 = 42 \end{aligned}$$

Example 10. Here we will use $u = \sin x$.

$$\int \sin^3 x \cos x dx = \int u^3 = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

3.2 Integration by Parts

Consider two functions $u(x), v(x)$. The product rule says that

$$\frac{d}{dx}(u(x) \cdot v(x)) = u'(x)v(x) + u(x)v'(x).$$

Integrating both sides of this equation now says that

$$u(x)v(x) = \int [u(x)v(x)]' = \int u'(x)v(x)dx + \int u(x)v'(x)dx = \int v(x)du + \int u(x)dv.$$

Rearranging, that is

$$\int u(x)dv = u(x)v(x) - \int v(x)du$$

or in more compact notation

$$\boxed{\int u dv = uv - \int v du} \quad (5)$$

Recall that

$$\begin{aligned} \frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \\ \frac{d}{dx} \arccos x &= \frac{-1}{\sqrt{1-x^2}} \Rightarrow - \int \frac{dx}{\sqrt{1-x^2}} = \arccos x + C \\ \frac{d}{dx} \arctan x &= \frac{1}{1+x^2} \Rightarrow \int \frac{dx}{1+x^2} = \arctan x + C \end{aligned}$$

Example 11. Now for a long list of examples with solutions!

1.

$$\int \arcsin x dx = x \arcsin x + \sqrt{1-x^2} + C.$$

Solution: With $u = \arcsin x, dv = dx$, so that $du = \frac{1}{\sqrt{1-x^2}}, v = x$ (using integration by parts - IBP) we find

$$\int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx.$$

Now, the substitution $y = 1 - x^2$ so that $dy = -2x dx$ or $x dx = -\frac{dy}{2}$ reveals the above as

$$x \arcsin x - \frac{1}{2} \int y^{-1/2} dy = x \arcsin x + \frac{1}{2} 2\sqrt{y} + C = x \arcsin x + \sqrt{1-x^2} + C.$$

2.

$$\int x^2 e^{-x} dx = -e^{-x}(x^2 + 2x + 2) + C$$

Solution: This will require two iterations of IBP. First we will have $u = x^2$ and $dv = e^{-x} dx$ so that $du = 2x dx$ and $v = -e^{-x}$. The first iteration becomes

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - \int (-2x) e^{-x} dx = -x^2 e^{-x} + 2 \int x e^{-x} dx.$$

For the second term we use $u = x, dv = e^{-x} dx$ so that $du = dx, v = -e^{-x}$ and the above becomes

$$-x^2 e^{-x} + 2 \left[-x e^{-x} + \int e^{-x} dx \right] = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

as required.

3.

$$\int e^{2x} \sin(3x) dx = \frac{13}{2} e^{2x} \left(2 \sin(3x) - \frac{3}{2} \cos(3x) \right) + C$$

Solution: This will also require two iterations of IBP, but has a slightly different solution.

Let $u = \sin(3x)$, $dv = e^{2x} dx$ so $du = 3 \cos(3x) dx$, $v = \frac{1}{2} e^{2x}$,

$$\int e^{2x} \sin(3x) dx = \frac{1}{2} \sin(3x) e^{2x} - \frac{3}{2} \int e^{2x} \cos(3x) dx$$

Now for the second term, use $u = \cos(3x)$, $dv = e^{2x} dx$ so that $du = -3 \sin(3x) dx$, $v = \frac{1}{2} e^{2x}$ and then the integral from the second term above becomes

$$\int e^{2x} \cos(3x) dx = \frac{1}{2} \cos(3x) e^{2x} + \frac{3}{2} \int e^{2x} \sin(3x) dx.$$

Notice that the original integral $y = \int e^{2x} \sin(3x) dx$ has reappeared. However, this doesn't mean we have gone in a circle since other terms have come out. Putting these together reads

$$\begin{aligned} y &= \frac{1}{2} \sin(3x) e^{2x} - \frac{3}{2} \int e^{2x} \cos(3x) dx \\ &= \frac{1}{2} \sin(3x) e^{2x} - \frac{3}{2} \left[\frac{1}{2} \cos(3x) e^{2x} + \frac{3}{2} \int e^{2x} \sin(3x) dx \right] \\ &= \frac{e^{2x}}{2} (\sin(3x) - \frac{3}{2} \cos(3x)) - \frac{9}{4} y \end{aligned}$$

or in other words,

$$(1 + \frac{9}{4})y = \frac{e^{2x}}{2} (\sin(3x) - \frac{3}{2} \cos(3x)) \Rightarrow \boxed{y = \frac{2}{13} e^{2x} (\sin(3x) - \frac{3}{2} \cos(3x))}$$

4.

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

Solution: Here again we will find a similar solution to the previous problem. Let I denote the value of interest, $\int \sec^3 x dx$, (for later when we need to rearrange). For IBP, use $u = \sec x$, $dv = \sec^2 x dx$ so that $du = \sec x \tan x dx$, $v = \tan x$. Now

$$\begin{aligned} I &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx = \sec x \tan x - I + \ln |\sec x \tan x| + C \end{aligned}$$

So, rearranging,

$$2I = \sec x \tan x + \ln |\sec x \tan x| + C \Rightarrow \boxed{I = \frac{1}{2} (\sec x \tan x + \ln |\sec x \tan x|) + C}$$

3.3 Trigonometric Integrals

Recall and remember the following trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \text{and} \quad \sin^2 x + \cos^2 x = 1$$

Now, we will

Evaluate: $\int \sin^m x \cos^n x dx$

by considering three different cases.

Solution:

(i) when n is odd;

Write $n = 2k + 1$, then by the substitution $u = \sin x$

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x dx \\ &= \int u^m (1 - u^2)^k du \end{aligned}$$

and this is easy to evaluate (begin a polynomial).

(ii) when m is odd;

Write $m = 2k + 1$ and the substitution $u = \cos x$

$$\int (\sin^2 x)^k \cos^n x \sin x dx = - \int (1 - u^2)^k u^n du$$

(iii) when both m and n are even;

The identity

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x)) \quad \text{or} \quad \sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

followed by

$$\sin x \sin y = \frac{1}{2}(\sin(x + y) + \sin(x - y)) \quad \text{or} \quad \cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y))$$

will bring one of the even powers down to either case (i) or case (ii).

and similarly,

Evaluate: $\int \tan^m x \sec^n x dx$

(i) If n is even, say $n = 2k$, then the substitution $u = \tan x$ and $du = \sec^2 x dx$ gives

$$\begin{aligned}\int \tan^m x (\sec^2 x)^k dx &= \int \tan^m \sec^{2k-2} x \sec^2 x dx \\ &= \int \tan^m (\sec^2 x)^{k-1} \sec^2 x dx \\ &= \int \tan^m (1 + \tan^2 x)^{k-1} \sec^2 x dx \\ &= \int u^m (1 + u^2)^{k-1} du\end{aligned}$$

(ii) If m is odd, then we will substitute $u = \sec x$ with $du = \sec x \tan x dx$ like so

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int (u^2 - 1)^k u^{n-1} du\end{aligned}$$

In any other case, we won't find a closed expression for the integral $\int \tan^m x \sec^n x dx$.

Example 12. Now for a list of examples with solutions!

1.

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int (1 - \cos^2 x) \cos^2 x \sin x dx, \quad u = \cos x, du = -\sin x dx \\ &= -\int (1 - u^2) u^2 du \\ &= -\int (u^2 - u^4) du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C\end{aligned}$$

2.

$$\begin{aligned}\int \cos^5 x dx &= \int (1 - \sin^2 x)^2 \cos x dx, \quad u = \sin x, du = \cos x dx \\ &= \int (1 - u^2)^2 du \\ &= \int (1 - 2u^2 + u^4) du \\ &= u - \frac{2u^3}{3} + \frac{u^5}{5} + C \\ &= \sin x - \frac{2 \sin^3 x}{3} + \frac{\sin^5 x}{5} + C\end{aligned}$$

3.

$$\begin{aligned}
 \int \sin^2 x \cos^2 x dx &= \int (\sin x \cos x)^2 dx \\
 &= \int \left(\frac{1}{2} \sin(2x)\right)^2 dx \\
 &= \frac{1}{4} \int \sin^2(2x) dx \\
 &= \frac{1}{4} \int \frac{1}{2} (1 - \cos(4x)) dx \\
 &= \frac{1}{8} \int (1 - \cos(4x)) dx \\
 &= \frac{1}{8} x + \frac{1}{8} \frac{\sin(4x)}{4} + C
 \end{aligned}$$

4.

$$\begin{aligned}
 \int \tan^3 x \sec x dx &= \int (\sec^2 x - 1) \sec^2 x \sec x \tan x dx, \quad u = \sec x, du = \sec x \tan x dx \\
 &= \int (u^2 - 1) u^2 du \\
 &= \frac{u^5}{5} - \frac{u^3}{3} + C
 \end{aligned}$$

5.

$$\begin{aligned}
 \int \tan^4 x \sec^4 x dx &= \int \tan^4 x (\tan^2 x + 1) \sec^2 x dx, \quad u = \tan x, du = \sec^2 x \\
 &= \int u^4 (u^2 + 1) du \\
 &= \frac{u^7}{7} + \frac{u^5}{5} + C
 \end{aligned}$$

3.4 Trigonometric Substitution

This method will apply to integrands containing $a^2 + x^2$, $a^2 - x^2$ or $x^2 - a^2$.

So, if

Integrand has	substitute	apply identity
$a^2 - x^2$	$x = a \sin x, -\pi/2 \leq x \leq \pi/2$	$1 - \sin^2 \theta = \cos^2 x\theta$ or $\cos^2 \theta = 1/2(1 + \cos 2\theta)$
$x^2 - a^2$	$x = a \sec x, -\pi/2 \leq x \leq \pi/2$ or $\pi \leq x \leq 3\pi/2$	$\sec^2 \theta - 1 = \tan^2 x\theta$
$a^2 + x^2$	$x = a \tan x, -\pi/2 \leq x \leq \pi/2$	$1 + \tan^2 \theta = \sec^2 x\theta$

Example 13. *These ideas will be best demonstrated through a list of examples*

1. Letting $x = \sin \theta$ so that on the interval $-\pi/2 \leq \theta \leq \pi/2$ we have $\theta = \arcsin x$ and $dx = \cos \theta d\theta$, then

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{1-x^2}} dx &= \int \frac{\sin^3 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta \\
 &= \int \frac{\sin^3 \theta}{|\cos \theta|} \cos \theta d\theta \\
 &= \int \sin^3 \theta d\theta \\
 &= \int (1 - \cos^2 \theta) \sin \theta d\theta, \quad \text{let } u = \cos \theta, du = -\sin \theta d\theta \\
 &= -\int (1 - u^2) du \\
 &= -u + \frac{u^3}{3} + C \\
 &= -\cos \theta + \frac{\cos^3 \theta}{3} + C \\
 &= -\sqrt{1-x^2} + \frac{\sqrt{1-x^2}^3}{3} + C
 \end{aligned}$$

2. The trigonometric substitution here is $x = \frac{3}{2} \tan \theta$ on $-\pi/2 \leq \theta \leq \pi/2$ so that $\theta = \arctan(2x/3)$ and $dx = \frac{3}{2} \sec^2 \theta d\theta$, then

$$\begin{aligned}
 \int \frac{1}{(4x^2+9)^2} dx &= \int \frac{1}{(4(\frac{3}{2} \tan \theta)^2 + 9)^2} (\frac{3}{2} \sec^2 \theta) d\theta \\
 &= \frac{1}{81} \cdot \frac{3}{2} \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta \\
 &= \frac{1}{54} \int \frac{1}{\sec^2 \theta} d\theta \\
 &= \frac{1}{54} \int \cos^2 \theta d\theta \\
 &= \frac{1}{54} \int \frac{1}{2} (1 + \cos(2\theta)) d\theta \\
 &= \frac{1}{108} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{1}{108} (\theta + \frac{1}{2} \sin(2\theta)) + C \\
 &= \frac{1}{108} (\arctan x + \frac{1}{2} 2 \sin \theta \cos \theta) + C \\
 &= \frac{1}{108} (\arctan x + \sin \theta \cos \theta) + C \\
 &= \frac{1}{108} (\arctan x + \frac{2x}{4x^2+9} \cdot \frac{3}{4x^2+9}) + C
 \end{aligned}$$

where the last line is found from the fact that $\tan \theta = \frac{2x}{3}$ so the hypotenuse is necessarily $4x^2+9$ hence $\cos \theta = \frac{3}{4x^2+9}$ and $\sin \theta = \frac{2x}{4x^2+9}$.

3. Here we use $x = \sin \theta$ on $\pi/2 \leq \theta \leq \pi/2$ so that $dx = \cos \theta d\theta$. Note that this is definite integral so we

have to change the bounds of integration. That is, when $x = 1/2, \theta = \pi/6$ and when $x = \sqrt{3}/2, \theta = \pi/3$.

$$\begin{aligned}
 \int_{1/2}^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}} dx &= \int_{\pi/6}^{\pi/3} \frac{\sin^3 \theta \cos \theta}{\sqrt{1-\sin^2 \theta}} d\theta \\
 &= \int_{\pi/6}^{\pi/3} \sin^3 \theta d\theta, \text{ let } u = \cos \theta \\
 &= \int_{\pi/6}^{\pi/3} (1 - \cos^2 \theta) \sin \theta d\theta, \text{ let } u = \cos \theta \\
 &= - \int_{u=\sqrt{3}/2}^{u=1/2} (1 - u^2) du \\
 &= \left[-u + \frac{u^3}{3} \right]_{\sqrt{3}/2}^{1/2} \\
 &= \left(\frac{1}{24} - \frac{1}{2} \right) - \left(\frac{3\sqrt{3}}{24} - \frac{\sqrt{3}}{2} \right) \\
 &= \frac{9\sqrt{3}-11}{24}
 \end{aligned}$$

4. Here we will use $x + 1 = 2 \sin \theta$ for $|\theta| \leq \pi/2$ so $\theta = \arcsin\left(\frac{x+1}{2}\right)$ and $dx = 2 \cos \theta d\theta$. Now

$$\begin{aligned}
 \int \frac{x}{\sqrt{3-2x-x^2}} dx &= \int \frac{x}{4-(x+1)^2} dx \\
 &= \int \frac{(2 \sin \theta - 1) 2 \cos \theta}{\sqrt{4(1-\sin^2 \theta)}} d\theta \\
 &= \int \frac{2 \sin \theta - 1}{\cos \theta} d\theta \\
 &= 2 \int \sin \theta d\theta - \int d\theta \\
 &= -2 \cos \theta - \theta + C \\
 &= -2\sqrt{4-(x+1)^2} - \arcsin\left(\frac{x+1}{2}\right) + C
 \end{aligned}$$

3.5 Integration with Partial Fractions

This method is used to solve a rational integrand (i.e. of the form $\frac{p(x)}{q(x)}$ for polynomials p and q).

Case 1: $\deg p \geq \deg q$, use long division.

Case 2: $\deg p < \deg q$, factor the denominator and split up with partial fractions.

Example 14. As usual, this is best demonstrated by example.

1.

$$\begin{aligned}
 \int \frac{x^3 + x^2 + x - 1}{x^2 + 2x + 2} dx &= \int (x-1) dx + \int \frac{x+1}{x^2 + 2x + 2} dx, \quad (\text{long division}) \\
 &= \frac{x^2}{2} - x + \frac{\ln|x^2 + 2x + 2|}{2} + C
 \end{aligned}$$

2.

$$\int \frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} dx = \int \frac{4x^2 - 3x - 4}{x(x-1)(x-2)} dx$$

and here we write

$$\begin{aligned}\frac{4x^2 - 3x - 4}{x(x-1)(x-2)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} \\ &= \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)} \\ &= \frac{(A+B+C)x^2 + (A+2B-C)x - 2A}{x(x-1)(x+2)}.\end{aligned}$$

This is the linear system

$$\begin{aligned}A + B + C &= 4, \\ A + 2B - C &= -3\end{aligned}$$

and

$$-2A = -4$$

which has solution $A = 2, B = -1, C = 3$. So,

$$\begin{aligned}\int \frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} dx &= \int \frac{4x^2 - 3x - 4}{x(x-1)(x+2)} dx \\ &= \int \frac{2}{x} dx - \int \frac{1}{x-1} dx + \int \frac{3}{x+2} dx \\ &= 2 \ln|x| - \ln|x-1| + 3 \ln|x+2| + C\end{aligned}$$

3. Using that

$$\frac{x^3 + 4x - 1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

has solution $A = 1, B = 0, C = 3, D = -4$, we find that

$$\begin{aligned}\int \frac{x^3 + 4x - 1}{x(x-1)^3} dx &= \int \frac{1}{x} dx + 3 \int \frac{1}{(x-1)^2} dx - 4 \int \frac{1}{(x-1)^3} dx \\ &= \ln|x| - 3(x-1)^{-1} + 2(x-1)^{-2} + C\end{aligned}$$

where we have used the substitution $u = x - 1, du = dx$ to integrate the second two terms.

4. Here, the partial fractions look slightly different because of the irreducible quadratic factor in the denominator

$$\frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}.$$

Notice, the linear $Cx + D$ above the quadratic denominator. This is solved the same way as above and we find that $A = 2, B = -1, C = 3, D = -2$, so

$$\begin{aligned}\int \frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2 + 1)} dx &= 2 \int \frac{1}{x} dx - \int \frac{1}{x^2} dx + \int \frac{3x - 2}{x^2 + 1} dx \\ &= 2 \ln|x| + \frac{1}{x} + 3 \int \frac{x}{x^2 + 1} - 2 \int \frac{1}{x^2 + 1} dx \\ &= 2 \ln|x| + \frac{1}{x} + \frac{3}{2} \ln|x^2 + 1| - 2 \arctan x + C\end{aligned}$$

3.6 Non-Elementary Functions

The *elementary functions* are polynomials, trigonometric, exponential, rational or any combination of these using $+$, $-$, $\times$, $\div$, $\exp$ or composition.

Consider, the function $F : \mathbb{N} \rightarrow \mathbb{N}$ defined by the rule that $F(n)$ = the n^{th} Fibonacci number. Is F an elementary function? The answer here is YES! Now, is every function elementary? NO.

For example, $f(t) := e^{t^2}$ is continuous on $\mathbb{R}$, so integrable. Then define $F(x) := \int_0^x e^{t^2} dt$ which has derivative $F'(x) = f(x)$ (by FTC I). It can be shown that $F(x)$ is not elementary, meaning you can't find a formula for $F(x)$.

4 Applications of integration

4.1 Areas between curves

Consider a region bounded above by $y = f(x)$ and below by $y = g(x)$.

Case 1:

$$\begin{aligned} A &= A_1 - A_2 \\ &= \int_a^b f(x)dx - \int_a^b g(x)dx \\ &= \int_a^b (f(x) - g(x))dx \end{aligned}$$

Case 2:

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_a^b f(x)dx + \left(- \int_a^b g(x)dx \right) \\ &= \int_a^b (f(x) - g(x))dx \end{aligned}$$

Case 3:

$$\begin{aligned} A &= A_1 - A_2 \\ &= - \int_a^b g(x)dx - \left(- \int_a^b f(x)dx \right) \\ &= \int_a^b (f(x) - g(x))dx \end{aligned}$$

Notice that in all three cases thus far, we have the area between f and g on $[a, b]$ when $f(x) \geq g(x)$ is given by the formula

$$\int_a^b (f(x) - g(x))dx.$$

Now, suppose that $f(x) \geq g(x)$ on $[a, c]$ and then $g(x) \geq f(x)$ on $[c, b]$. Then

Case 4:

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx \\ &= \int_a^b (f(x) - g(x))dx \end{aligned}$$

Example 15. Find the area of the region bounded by...

1. $Y = x$ and $y = 6 - x^2$;

Solution: Since there is no given interval on which to find the area, we must compute where these two curves meet. That is, solve $x = y = 6 - x^2$ which is equivalent to $0 = (x + 3)(x - 2)$ so that these curves meet at $x = -3$ and $x = 2$. On this interval we find that $6 - x^2 \geq x$ so the area between them is computed by

$$\int_{-3}^2 (6 - x^2 - x)dx = \left[6x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{-3}^2 = \frac{125}{6}$$

2. $y = \frac{x}{2}$ and $y^2 = 8 - x$. These curves meet when $\frac{x^2}{4} = 8 - x$ which has solutions $x = -8$ and $x = 4$

Solution 1: Here we integrate along the x -axis as follows;

$$A = \int_{-8}^4 \left(\frac{x}{2} - (-\sqrt{8-x}) \right) dx + 2 \int_4^8 \sqrt{8-x} dx$$

Solution 2: If we tilt our heads, this is actually easier done by integrating along the y -axis. The intersection points are $(-8, -4)$ and $(4, 2)$ and the area is given by

$$\int_{-4}^2 (8 - y^2 - 2y)dy = \left(8y - \frac{y^3}{3} - y^2 \right)_{-4}^2$$

4.2 Arc Length of a Parameterized Curve

Let $\mathcal{C}$ be the smooth curve obtained by graphing $y = f(x)$ over the interval $[a, b]$. For the arc length of $\mathcal{C}$ we apply the following simple recipe.

1. Subdivide the interval (a, b) into n equal subintervals of length $\Delta x = \frac{b-a}{n}$
2. Let $a = x_0, \dots, b = x_n$
3. Let p_i be the point $(x_i, f(x_i))$ on $\mathcal{C}$ and $d(p_{i-1}, p_i)$ be the Euclidean distance between the points p_{i-1} and p_i . That is,

$$d(p_{i-1}, p_i) = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = (x_i - x_{i-1}) \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)^2} = \Delta x \sqrt{1 + f'(x_i^*)^2}$$

where the last equality holds by the *Mean Value Theorem*.

Hence, now the approximate length of $\mathcal{C}$ is given by

$$\sum_{i=1}^n \sqrt{1 + f'(x_i^*)^2} \Delta x$$

another Riemann sum! Finally, the *arc length* of $\mathcal{C}$ is given by

$$\boxed{\text{Length}(\mathcal{C}) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + f'(x_i^*)^2} \Delta x = \int_a^b \sqrt{1 + f'(x)^2} dx} \quad (6)$$

Example 16. Find the length of the curve $\mathcal{C}$ defined by $y = x^{3/2}$ on the interval $[0, 5]$.

Solution:

$$\begin{aligned} \text{Length}(\mathcal{C}) &= \int_0^5 \sqrt{1 + y'(x)^2} dx \\ &= \int_0^5 \sqrt{1 + \frac{9}{4}x} dx \\ &= \frac{1}{2} \int_0^5 \sqrt{4 + 9x} dx = \frac{335}{27} \end{aligned}$$

5 Calculating Volumes

5.1 Cross sectional method

The cross sectional method is often the most practical method used to compute volumes of *solids of revolution* (a solid obtained by revolving a region bounded by $y = f(x)$ over $[a, b]$ about the x - or y -axis).

Principle behind the cross section method

1. subdivide $[a, b]$ into n equal sub intervals, each of length $\Delta x = \frac{b-a}{n}$
2. Let $A(x_i)$ denote the area of the cross section of the solid at x_i
3. Then $A(x_1), A(x_2), \dots$ is a sequence of areas of cross sections, each a function of x_i .
4. If we then keep $A(x)$ constant over the interval (x_i, x_{i+1}) we obtain an approximation $V_n(x_i) = A(x_i)\Delta x$
5. We now have a sequence of volumes $V_n(x_1), V_n(x_2), \dots$
6. The n^{th} approximation of the volume of this solid is then

$$V_n = \sum_{i=1}^n V_n(x_i) = \sum_{i=1}^n A(x_i)\Delta x$$

which is a Riemann sum.

7. So, theoretically, the actual volume of the solid is

$$V = \int_a^b A(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i)\Delta x$$

Example 17. Using the cross sectional method to calculate the volume of a square based pyramid, with base $b \times b$ and height h .

Solution: The side length at x can be found using similar triangles as $s(x) = \frac{xb}{h}$, so then the area at x is then $A(x) = s(x)^2 = \frac{x^2 b^2}{h^2}$. Now, the volume can be computed by

$$V = \int_0^h A(x) dx = \frac{b^2}{h^2} \int_0^h x^2 dx = \frac{b^2}{h^2} \left[\frac{x^3}{3} \right]_0^h = \frac{b^2 h}{3}$$

Note that this method would also work using the y -axis and integrating with respect to y .

The above example was not a solid of revolution. However, there is a nice formula for the cross sectional area (and hence the volume) of a solid of revolution.

Notice, that when $y = f(x)$ is revolved about, say, the x -axis, the cross sectional area is a circle, of radius $f(x)$ centred at the x axis. So then $A(x) = \pi f(x)^2$ (the area of a circle of radius $f(x)$). Hence,

$$\boxed{V = \int_a^b \pi f(x)^2 dx} \quad (7)$$

If this was about the y -axis then $V = \int_c^d \pi g(y) dy$.

Example 18. All of the following are solids of revolution.

1. Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Solution: Here we use the function $y = \sqrt{r^2 - x^2}$ which is the top half of the circle to be revolved. So then the volume is achieved by integrating $A(x) = \pi y^2$ along the interval $[-r, r]$. That is,

$$V = \int_{-r}^r \pi (\sqrt{r^2 - x^2})^2 dx = \pi \int_{-r}^r (r^2 - x^2) dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r = \frac{4}{3} \pi r^2$$

2. Find the volume of a cone with height h and radius r .

Solution: This will be done along the y -axis, since standing cones make more sense than sideways cones. The relationship between x and y here is that $y = \frac{h}{r}x$ (which can be thought of as the slope of a cone whose tip is at the origin). So the radius at y is $x = \frac{r}{h}y$ and the volume is then given by

$$V = \pi \int_0^h \left(\frac{r}{h} y \right)^2 dy = \pi \frac{r^2}{h^2} \int_0^h y^2 dy = \pi \frac{r^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3} \pi r^2 h$$

Now, we are ready for something slightly more involved.

Example 19. Find the volume of the solid formed by revolving the region bounded by $y^2 = x$ and $y = x^3$ about the (i) x -axis, (ii) y -axis, and (iii) the $x = -1$ axis.

Notice beforehand that the intersection occurs at $(0, 0)$ and $(1, 1)$. So,

- (i) Revolving about the x -axis,

$$V = \int_0^1 \pi (\sqrt{x^2} - (x^3)^2) dx = \pi \left[\frac{x^2}{2} - \frac{x^7}{7} \right]_0^1 = \frac{5}{14} \pi$$

- (ii) About the y -axis,

$$V = \int_0^1 \pi ((\sqrt[3]{y})^2 - (y^2)^2) dy = \pi \left[\frac{3}{5} y^{5/3} - \frac{y^5}{5} \right]_0^1 = \frac{2}{5} \pi$$

- (iii) About $x = -1$ axis, we need to readjust the integrand.

$$\begin{aligned} V &= \int_0^1 \pi ((\sqrt[3]{y} + 1)^2 - (y^2 + 1)^2) dy \\ &= \pi \int_0^1 (y^{2/3} + 2y^{1/3} + 1 - y^4 - 2y^2 - 1) dy \\ &= \pi \left[\frac{2}{5} y^{5/2} + \frac{6}{4} y^{4/3} - \frac{y^5}{5} - \frac{2}{3} y^3 \right]_0^1 \\ &= \frac{37}{30} \pi \end{aligned}$$

5.2 Cylindrical Shell Method

The volume of a shell is equal to the outer volume minus the inner volume. Mathematically if $r < R$ represent the inner and outer radius of the cylinder respectively, then

$$\begin{aligned} V &= \text{outer} - \text{inner} \\ &= \pi R^2 h - \pi r^2 h \\ &= \pi h (R^2 - r^2) \\ &= 2\pi h \frac{R+r}{2} (R-r) \\ &= 2\pi h R^* t \end{aligned}$$

where $R^* = \frac{R+r}{2}$ is the average radius and $t = R - r$ is the thickness of the shell.

Steps for this method:

1. Suppose we have a region bounded by $y = f(x)$ and suppose it is revolved about the y -axis,
2. Subdivide the interval $[a, b]$ into n equal subintervals $\Delta x = \frac{b-a}{n}$,
3. Let $V(x_i)$ represent the volume of the cylindrical shell with outer radius x_i , inner radius x_{i-1} and height $f(x_i)$,
4. then we have a sequence $V(x_1), V(x_2), \dots$ of volumes of shells which fill our solid so the approximate volume is

$$\sum_{i=1}^n V(x_i) = \sum_{i=1}^n 2\pi f(x_i) x_i^* \Delta x, \text{ where } x_i^* = \frac{x_i + x_{i-1}}{2}$$

which is approximately

$$\sum_{i=1}^n 2\pi x_i f(x_i) \Delta x$$

a Riemann sum! Then the volume is given by

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi x_i f(x_i) \Delta x = 2\pi \int_a^b x f(x) dx \quad (8)$$

Note: This is most often used when there is a hole punched in the solid.

Example 20. Find the volume of the solid that remains after a hole of radius a has been bored through the centre of a sphere of radius $b (> a)$.

Solution:

$$\begin{aligned} V &= 2 \int_a^b 2\pi x \sqrt{b^2 - a^2} dx, \quad \text{let } x = b \sin \theta, dx = b \cos \theta \\ &= 4\pi \int_{x=a}^{x=b} b^2 \sin \theta \sqrt{b^2(1 - \cos^2 \theta)} b \cos \theta d\theta \\ &= 4\pi b^3 \int_{x=a}^{x=b} \sin \theta \cos^2 \theta d\theta, u = \cos \theta \\ &= 4\pi b^3 \int_{x=a}^{x=b} u^2 du \\ &= 4\pi b^3 \left[\frac{u^3}{3} \right]_{x=a}^{x=b} \end{aligned}$$

Now, with $u = \cos \theta = \frac{\sqrt{b^2 - x^2}}{b}$ we have

$$4\pi b^3 \left[\frac{u^3}{3} \right]_{x=a}^{x=b} = 4\pi b^3 \left[\frac{(b^2 - x^2)^{3/2}}{b^3} \right]_{x=a}^{x=b} = \frac{4}{3}\pi(b^2 - a^2)^{3/2}$$

Example 21. Using the method of cylindrical shells, find the volume obtained by revolving the region bounded by $y = x^2$ and $y^2 = x$ about the y -axis.

Solution:

$$V = \int_0^1 2\pi x (\sqrt{x} - x^3) dx = 2\pi \int_0^1 (x^{3/2} - x^4) dx = 2\pi \left[\frac{2}{5}x^{5/2} - \frac{x^5}{5} \right]_0^1 = \frac{2}{5}\pi$$

In general we have obtained the following two formulas

$$V = \int_a^b \pi f(x) dx \text{ and } V = \int_a^b 2\pi x f(x) dx$$

6 Numerical Approximation

This is used to approximate definite integrals when they cannot be integrated.

There are 3 common methods:

1. *Midpoint Rule:* Given $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$, recall that we had a choice of sample points x_i in the interval (x_{i-1}, x_i) and we have just been using the right hand side of the interval. Now, consider

$$R_n = \sum_{i=1}^n f(x_i) \Delta x \quad - \quad \text{The right-hand approximation}$$

$$L_n = \sum_{i=1}^n f(x_{i-1}) \Delta x \quad - \quad \text{The left-hand approximation}$$

and finally

$$M_n = \sum_{i=1}^n f(m_i) \Delta x \quad - \quad \text{The Midpoint approximation}$$

where $m_i = \frac{x_{i-1} + x_i}{2}$

2. *Trapezoidal Rule:* Here we use

$$T_n = \frac{L_n + R_n}{2} = \frac{1}{2} \left[\sum_{i=1}^n f(x_{i-1}) \Delta x + \sum_{i=1}^n f(x_i) \Delta x \right] = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

3. *Simpson's Rule:* Here we use $2n$ sample points $x_1, x_2, \dots, x_n, \dots, x_{2n}$, but our intervals remain of size $\Delta x = \frac{b-a}{n}$

$$S_{2n} := \frac{1}{3} [2M_n + T_n]$$

Now,

$$M_n = 2\Delta x \left[\sum_{i=1}^n f(x_{2i-1}) \right]$$

because midpoints here are the odd indexed sample points and

$$T_{2n} = 2 \frac{\Delta x}{2} (f(x_0) + 2f(x_2) + \dots 2f(x_{2n-2}) + f(x_{2n}))$$

So,

$$\begin{aligned} S_{2n} &= \frac{\Delta x}{3} \left[4 \sum_{i=1}^n f(x_{2i-1}) + (f(x_0) + 2f(x_2) + \dots 2f(x_{2n-2}) + f(x_{2n})) \right] \\ &= \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{2n-1}) + f(x_{2n})) \end{aligned}$$

Example 22. Approximate the definite integral $\int_0^3 x^2 dx$ with $n = 6$ (i.e. so that $\Delta x = \frac{3-0}{6} = .5$ using; (i) midpoint rule, (ii) trapezoid rule, (iii) Simpson's rule.

Solution:

(i) For this, we need the following table of values:

n	m_i	$f(m_i)$
1	.25	0.0625
2	.75	0.5625
3	1.25	1.5625
4	1.75	3.0625
5	2.25	5.0625
6	2.75	7.5625

So,

$$M_n = \sum_{i=1}^6 f(m_i) \Delta x = \frac{1}{2} (19.875) = 9.9375$$

(ii)

$$\begin{aligned} T_n &= \frac{\Delta x}{2} (f(0) + 2f(0.5) + 2f(1) + \dots + 2f(5.5) + f(6)) \\ &= \frac{1}{4} (0^2 + 2(0.5)^2 + \dots + 3^2) \\ &= 9.125 \end{aligned}$$

(iii)

$$S_n = \frac{\Delta x}{3} (0^2 + 4(0.5)^2 + 2(1)^2 + 4(1.5)^2 + 2(2)^2 + 4(2.5)^2 + 3^2) = \frac{1}{6} (54) = 9$$

6.1 Error bounds on approximations

Here we will just provide the error bounds without examining the required analysis.

For M_n ; the error bound is

$$|E(M_n)| \leq \frac{k(b-a)^3}{24n^2}$$

where $k = \max_{x \in (a,b)} \{f''(x)\}$.

For T_n : Error is

$$|E(T_n)| \leq \frac{k(b-a)^3}{12n^2}, \quad k = \max_{x \in (a,b)} \{f''(x)\}$$

For S_n : Error is

$$|E(S_n)| \leq \frac{k(b-a)^5}{180n^4}, \quad k = \max_{x \in (a,b)} \{f^{(4)}(x)\}.$$

Example 23. Estimate the error bound when using T_{10} or S_{10} approximation for $\int_1^2 \frac{1}{x} dx$.

Solution: Note that

$$\frac{d^2}{dx^2} \left(\frac{1}{x} \right) = \frac{2}{x^3}$$

and

$$\frac{d^4}{dx^4} \left(\frac{1}{x} \right) = \frac{d^2}{dx^2} \left(\frac{2}{x^3} \right) = \frac{d}{dx} \left(\frac{-6}{x^4} \right) = \frac{24}{x^5}$$

so that the constant k for T_{10} is

$$k_T = \max_{[1,2]} \left(\frac{2}{x^3} \right) = 2$$

and for S_{10} is

$$k_S = \max_{[1,2]} \left(\frac{24}{x^5} \right) = 24.$$

Now,

$$|E(T_{10})| \leq \frac{k_T(2-1)^3}{12(10)^2} = \frac{1}{600}$$

and

$$|E(S_{10})| \leq \frac{k_S(2-1)^5}{180(10)^4} = \frac{24}{1800000}.$$

7 Improper Integrals

The following integrals

$$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx \text{ OR } \int_{-\infty}^a f(x)dx = \lim_{b \rightarrow -\infty} \int_b^a f(x)dx$$

are called *improper*. Similarly,

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

is improper (provided, both limits exist).

Warning: Do not do $\int_{-\infty}^\infty f(x)dx = \lim_{t \rightarrow \infty} \int_{-t}^t f(x)dx$, since one may not converge.

Example 24. Evaluate the improper integral $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$.

Solution:

$$\begin{aligned} \int_{-\infty}^\infty \frac{1}{1+x^2} dx &= \lim_{b \rightarrow -\infty} \int_b^0 \frac{1}{1+x^2} dx + \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+x^2} dx \\ &= \lim_{b \rightarrow -\infty} [\arctan x]_b^0 + \lim_{t \rightarrow \infty} [\arctan x]_0^t \\ &= \left(0 - \frac{-\pi}{2} \right) + \left(\frac{\pi}{2} - 0 \right) \\ &= \pi \end{aligned}$$

7.1 Improper integrals of another type

Suppose the function f is defined only on the interval $(a, b]$ with no specified value at a . Then we can still define the definite integral ‘improperly’ as

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^-} \int_t^b f(x)dx.$$

The same can be done when $f(x)$ is defined on $[a, b)$, or even if, say $f(x)$ is only defined on $[a, c) \cup (c, b]$ (with a hole in the middle of the interval), then

$$\int_a^b f(x)dx = \lim_{t \rightarrow c^-} \int_a^t f(x)dx + \lim_{t \rightarrow c^+} \int_t^b f(x)dx$$

Note that the above integral will only converge if both limits exist.

Example 25.

Proposition 1. [Comparison test for improper integrals] Suppose that $f(x) \geq g(x)$ on $[a, \infty)$. If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ also converges.

Furthermore, if $\int_a^\infty g(x)dx$ diverges, then so does $\int_a^\infty f(x)dx$.

Example 26. Does $\int_0^\infty e^{-x^2} dx$ converge?

Solution: We have $x^2 \geq x$ on $[1, \infty]$ so $e^{x^2} \geq e^x$ on $[1, \infty)$ or $\frac{1}{e^{x^2}} \leq \frac{1}{e^x}$. So

$$\begin{aligned} \int_0^\infty e^{-x^2} dx &= \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx \\ &\leq \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x} dx \\ &= \int_0^1 e^{-x^2} dx + \lim_{t \rightarrow \infty} \int_1^t e^{-x} dx \\ &= \int_0^1 e^{-x^2} dx + \lim_{t \rightarrow \infty} [-e^{-t} - (-e^{-1})] \\ &= \int_0^1 e^{-x^2} dx + e < \infty \end{aligned}$$

so our integral converges by comparison test.

8 Differential Equations

A *differential equation* (DE) is an equation which contains a derivative expression. The *order* of a DE is the highest derivative which appears in the equation.

Example 27. $xy''(x) = \sin x e^{y(x)}$ is order 2.

Another, simpler, DE is $y'(x) = \sin x + e^x$. This is easy to solve (integrate both sides), $y(x) = -\cos x + e^x + C$ is a solution.

Lets see a more exciting example.

Example 28. Let $P(t)$ be a population function (with t for time). After studying the population over time, and discovering a proportionality amongst the $P(t)$ and its derivative. That is to say $P'(t) = kP(t)$, for

some constant k . The goal now, is to obtain a formula for $P(t)$. Rearranging slightly is

$$\begin{aligned}\frac{P'(t)}{P(t)} &= k \Rightarrow \int \frac{P'(t)}{P(t)} dt = \int k dt \\ &\Leftrightarrow \ln |P(t)| = kt + C \\ e^{\ln P(t)} &= e^{kt+C} \\ P(t) &= e^{kt} e^C \\ P(t) &= C' e^{kt}\end{aligned}$$

for some positive constant C that is determined by specified initial conditions (such as $P(0) = 100$).

Example 29. Verify that $y = x^2 \ln x$ is a solution to $x^2 y'' - 3xy' + 4y = 0$ in the region $x > 0$.

Solution: With $y = x^2 \ln x$, we compute $y' = 2x \ln x + x$ and $y'' = 2 \ln x + 2 + 1$. Substituting into the DE, we find

$$\begin{aligned}x^2 y'' - 3xy' + 4y &= x^2(2 \ln x + 3) - 3x(2x \ln x + x) + 4x^2 \ln x \\ &= 2x^2 \ln x + 3x^3 - 6x^2 \ln x - 3x^2 + 4x^2 \ln x \\ &= 0\end{aligned}$$

A particular solution to a DE is one without arbitrary constants. That is, any particular function satisfying the DE. The solution $y = x^2 \ln x$ from the previous example is a particular solution.

Example 30. Suppose we want a solution to $y' + 2y = e^{-x}$ with $y(0) = 3$ [This is called an initial value problem].

Check that $y = e^{-x} + Ce^{-2x}$ is a general solution. Indeed, $y' = -e^{-x} - 2Ce^{-2x}$ and

$$-e^{-x} - 2Ce^{-2x} + 2(e^{-x} + Ce^{-2x}) = 0.$$

Now since $y(0) = 3$, we can solve for the constant C as

$$3 = e^0 + Ce^0 = 1 + C \Rightarrow C = 2$$

so that $y = e^{-x} + 2e^{-2x}$ is a particular solution to the initial value problem (IVP).

8.1 Separable Differential Equations

If a DE can be expressed in the form $y' = g(x)f(y)$, we say it is *separable*. Equivalently

$$\frac{1}{f(y)} \frac{dy}{dx} = g(x) \Leftrightarrow \frac{1}{f(y)} dy = g(x) dx.$$

A separable DE is solved by integrating both sides of this rearrangement.

$$\int \frac{1}{f(y)} dy = \int g(x) dx,$$

but this requires some justification: If we are given a DE of the form $N(y)dy = M(x)dx$ or equivalently $N(y)\frac{dy}{dx} - M(x) = 0$. Now let $H_1(y)$ be an anti-derivative of $N(y)$ and $H_2(x)$ an anti-derivative of $M(x)$. This equation is then equivalent to the following:

$$\begin{aligned}H_1'(y) \frac{dy}{dx} - H_2'(x) &= 0 \\ \frac{d}{dx}(H_1(y)) - H_2'(x) &= 0 \\ = \frac{d}{dx}(H_1(y) - H_2(x)) &= 0\end{aligned}$$

Now, this implies that $H_1(y) - H_2(x) = K$ (is constant) and we can find $y(x)$ from $H_2(y)$.

Example 31. Is $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$ separable? If so, solve it.

Solution: Yes, it is solvable and it can be re-expressed as

$$2(y-1)dy = (3x^2 - 4x + 2)dx \Rightarrow 2 \int (y-1)dy = \int (3x^2 - 4x + 2)dx.$$

This integrates to

$$y^2 - 2y = x^3 + 2x^2 + 2x + C.$$

Now, although this is separable and we managed to integrate it, we find that the solution cannot be explicitly expressed as $y = \dots$. In attempt to extract an explicit solution, we treat this as a quadratic equation in y .

$$y^2 - 2y - (x^3 + 2x^2 + 2x + C) = 0$$

which has solutions,

$$y = \frac{2 \pm \sqrt{4 - 4(-1)(x^3 + 2x^2 + 2x + C)}}{2} = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 1 + C}.$$

If we had been given the IVP of $y(0) = -1$, then this would imply that the solution be $y = 1 - \sqrt{x^3 + 2x^2 + 2x + 1 + C}$.

8.2 Linear Differential Equations

A linear differential equation is most generally expressed as

$$y' + p(x)y = q(x) \quad (9)$$

Theorem 6. The solution to the linear differential equation (9) is

$$y = \frac{1}{I(x)} \int I(x)q(x)dx$$

where $I(x) = e^{\int p(x)dx}$.

This function $I(x)$ is called the integrating factor and is a function satisfying that

$$I(x)(y' + p(x)y) = (I(x)y)'$$

Proof. First, verifying that $I(x) = e^{\int p(x)dx}$ satisfies the above claimed equation. Indeed, expanding the right by the product rule, we have

$$I(x)(y' + p(x)y) = (I(x)y)' = I'(x)y + I(x)y' \Leftrightarrow I(x)p(x)y = I'(x)y$$

so that, provided $I(x) \neq 0$, $p(x) = \frac{I'(x)}{I(x)}$. Integrating this says that

$$\int p(x)dx = \int \frac{I'(x)}{I(x)}dx \Leftrightarrow \ln |I(x)| = \int p(x)dx \Leftrightarrow I(x) = e^{\int p(x)dx}.$$

Now, that this holds, the left hand side of our DE is expressed as $(I(x)y)'$ and now reads, $(I(x)y)' = q(x)$. Hence, integrating this, $I(x)y = \int q(x)dx$ so $y = \frac{1}{I(x)} \int I(x)q(x)dx$ as claimed. $\square$

Note that this method of solving a DE requires it be of first order (i.e. linear).

Example 32. 1. Solve the IVP $y' - 2xy = 1$ with $y(0) = 1$.

Solution: The integrating factor here is $I(x) = e^{-\int 2x dx} = e^{-x^2}$, so

$$y = \frac{1}{e^{-x^2}} \int e^{-x^2} dx = e^{x^2} \left(\int_0^x e^{-t^2} dt + C \right) = e^{x^2} \int_a^x e^{-t^2} dt + Ce^{x^2}$$

and with $y(0) = 1$ we find C as

$$1 = e^0 \int_0^0 e^{-t^2} dt + e^0 C \Rightarrow C = 1$$

2. Solve the IVP $(y - x \sin x)dx + xdy = 0$ with $y(\pi) = 3$.

Solution: This is equivalent to $y = x \sin x + xy' = 0$ or, better, $y' + \frac{1}{x}y = \sin x$. For the integrating factor $I(x) = e^{\int \frac{1}{x} dx} = e^{\ln |x|} = |x|$. Now,

$$\begin{aligned} y &= \frac{1}{|x|} \int |x| \sin x dx, \\ &= \frac{1}{x} \left[-x \cos x + \int \cos x dx \right], \quad \text{IBP with } u = x, dv = \sin x dx \\ &= -\cos x + \frac{\sin x}{x} + \frac{C}{x} \end{aligned}$$

9 Sequences and Series

Recall, the limit of a sequence: A sequence $\{x_n\}$ converges to L if for all $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n > k$.

Example 33. $a_n = (-1)^n \frac{\cos n}{n^2}$ and $|(-1)^n \frac{\cos n}{n^2}| \leq \frac{1}{n^2} \rightarrow 0$ and by squeeze theorem, $a_n \rightarrow 0$.

An *increasing (decreasing) sequence* is one such that $a_{n+1} \geq (\leq) a_n$ for all $n \geq 1$. A *monotonic sequence* is one which is either strictly increasing or decreasing.

Theorem 7 (Monotone sequence theorem). A *bounded monotonic sequence always has a limit (i.e. will always converge)*.

9.1 Series

Let a_n be a sequence, then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ is an *infinite series*. The n^{th} *partial sum* of a series is $S_n = \sum_{k=1}^n a_k$. To evaluate an infinite series we use the limit of the partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

If the limit exists, series converges. If limit does not exist, series *diverges*.

Warning: When discussing series, say $\sum_{n=1}^{\infty} c_n$, there are two sequences in play. We have the sequence itself c_n and the partial sums S_n , so remember

$$\sum_{n=1}^{\infty} = \lim_{n \rightarrow \infty} S_n.$$

Example 34.

$$c_i = i \Rightarrow \sum n = 1^\infty n = \lim_{n \rightarrow \infty} \sum_{k=1}^n k = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2} = \infty, \text{ (diverges)}$$

Consider the *geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots + ar^n + \dots$$

The partial sums are given by

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

and multiplying by r says,

$$rS_n = ar + ar^2 + \dots + ar^n.$$

Subtracting these, we have

$$S_n - rS_n = a - ar^n \Leftrightarrow S_n(1 - r) = a(1 - r^n)$$

so that

$$S_n = \frac{1(1 - r^n)}{1 - r}.$$

Now, when does a geometric series converge? We must have that $\lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r}$ exists, which means, in order for a geometric series to converge we must have $|r| < 1$.

Furthermore, if $|r| < 1$, then the limit can be computed exactly as $\lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}$. That is, the sum a geometric series is

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ \infty, & |r| \geq 1 \end{cases}$$

Example 35. Express the repeating decimal $1.2\overline{31}$ as an infinite series and then as a fraction.

Solution: We have

$$1.2\overline{31} = 1.2 + \frac{31}{10^3} + \frac{31}{10^3} \cdot \frac{1}{10^2} + \frac{31}{10^3} \cdot \frac{1}{(10^2)^2} + \dots$$

Letting $a = \frac{31}{10^3}$ and $r = \frac{1}{10^2} < 1$, so that

$$1.2\overline{31} = 1.2 + \sum_{n=1}^{\infty} ar^{n-1} = \frac{12}{10} + \frac{31}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}} = \frac{1219}{990}.$$

Theorem 8. Suppose $\sum_{n=1}^{\infty} c_n = L$ and $\sum_{n=1}^{\infty} b_n = M$ and let $k \in \mathbb{R}$. Then,

$$\sum_{n=1}^{\infty} kc_n = kL$$

and

$$\sum_{n=1}^{\infty} (c_n + b_n) = L + M.$$

Note that this theorem requires that both converge.

Theorem 9. If the series $\sum_{n=1}^{\infty} c_n$ converges, then the sequence c_n must converge to 0.

Proof. Given that $\sum_{n=1}^{\infty} c_n = L$, consider the sequence S_n of partial sums. Observe, that for each n , $c_n = S_n - S_{n-1}$ and now,

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} S_n - S_{n-1} = L - L = 0.$$

□

9.2 Divergence test

Suppose that $\sum_{n=1}^{\infty} c_n$ is a series. If $\lim_{n \rightarrow \infty} c_n \neq 0$ then the series diverges. This is just the converse of the previous result. Otherwise, divergence test provides no information.

Example 36. Discuss the convergence/divergence of the series

$$1. \sum_{n=1}^{\infty} \left(\frac{3}{n} + \frac{5}{4^n} \right).$$

Solution:

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{3}{n} + \frac{5}{4^n} \right) &= \sum_{n=1}^{\infty} \frac{3}{n} + \sum_{n=1}^{\infty} \frac{5}{4^n} \\ &= 3 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges}} + \underbrace{\sum_{n=1}^{\infty} \frac{5}{4} \left(\frac{1}{4} \right)^{n-1}}_{\text{converges (geometric } r=1/4 \text{)}}\end{aligned}$$

Therefore, the series diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)}$

Solution: By partial fractions, we find

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} &= \sum_{n=1}^{\infty} \left(\frac{-1}{2(2n+1)} + \frac{1}{2(2n-1)} \right) \\ &= \lim_{n \rightarrow \infty} -\frac{1}{6} + \frac{1}{2} - \frac{1}{10} + \frac{1}{6} - \frac{1}{14} + \frac{1}{10} + \cdots + \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} - \frac{1}{2(2n-1)} = \frac{1}{2}\end{aligned}$$

This is a telescoping series.

Theorem 10 (Integral test for convergence of a series). Let $\sum_{n=1}^{\infty} c_n$ be a series and suppose $f(x)$ is a function on $[1, \infty)$ such that

1. $f(x) > 0$ on $[1, \infty)$.
2. $f(x)$ is continuous on $[1, \infty)$
3. $f(x)$ is non-increasing on $[1, \infty)$ (i.e. $f'(x) \leq 0$)
4. $f(n) = c_n$ for all n .

Then,

$$\sum_{n=1}^{\infty} c_n \text{ converges} \Leftrightarrow \int_1^{\infty} f(x) dx \text{ converges.}$$

Proof. Suppose that $\sum_{n=1}^{\infty} c_n$ converges and show that $\int_1^{\infty} f(x) dx$ converges. Certainly

$$S_n = c_1 + c_2 + \cdots + c_n \geq \int_1^n f(x) dx$$

so

$$S = \lim_{n \rightarrow \infty} S_n \geq \int_1^{\infty} f(x) dx$$

which implies convergence.

Suppose now that $\int_1^{\infty} f(x) dx$ converges and show the converse. Observe that

$$c_2 + c_3 + \cdots + c_n \leq \int_1^{\infty} f(x) dx$$

so

$$S_n = c_1 + c_2 + \cdots + c_n \leq c_1 + \int_1^{\infty} f(x) dx$$

and hence $\lim_{n \rightarrow \infty} S_n \leq c_1 + \int_1^{\infty} f(x) dx < \infty$ so converges. □

9.3 Estimating the value of a series

Suppose that $\sum_{n=1}^{\infty} c_n$ is a series proved to be convergent by the integral test (say $\sum_{n=1}^{\infty} c_n = S$).

We would like to approximate S by the partial sums S_n and determine what the error will be. Let $R_n = S - S_n$ be the *remainder* (Note that, we are assuming that our c_n 's are positive so that the partial sums are an increasing sequence). So, also

$$R_n = S - S_n = (c_1 + c_2 + \cdots + c_n + \cdots) - (c_1 + c_2 + \cdots + c_n) = c_{n+1} + c_{n+2} + \cdots$$

and we claim that

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

for $f(n) = c_n$.

First, it is easy to see that $R_n = c_{n+1} + c_{n+2} + \cdots \leq \int_n^{\infty} f(x)dx$. Now, also $R_n \geq \int_{n+1}^{\infty} f(x)dx$ since these rectangles are an over approximation of the integral.

Now,

$$\begin{aligned} \int_{n+1}^{\infty} f(x)dx &\leq R_n \leq \int_n^{\infty} f(x)dx \\ S_n + \int_{n+1}^{\infty} f(x)dx &\leq S_n + R_n \leq S_n + \int_n^{\infty} f(x)dx \\ S_n + \int_{n+1}^{\infty} f(x)dx &\leq S \leq S_n + \int_n^{\infty} f(x)dx \end{aligned}$$

So we know that S lies in the interval $\left[S_n + \int_{n+1}^{\infty} f(x)dx, S_n + \int_n^{\infty} f(x)dx \right]$ and the error cannot be more than the length of the interval. That is,

$$|\text{Error}| \leq (S_n + \int_n^{\infty} f(x)dx) - (S_n + \int_{n+1}^{\infty} f(x)dx) = \int_n^{\infty} f(x)dx - \int_{n+1}^{\infty} f(x)dx = \int_n^{n+1} f(x)dx$$

Example 37. Use S_{10} to help calculate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution: With a calculator we find $S_{10} = 1.082036583$. Notice that $\frac{1}{x^4}$ is continuous, decreasing on $[1, \infty)$ and matches the values of the sequence. So, a good approximation is the midpoint of the interval $\left[S_{10} + \int_{11}^{\infty} \frac{1}{x^4}dx, S_{10} + \int_{10}^{\infty} \frac{1}{x^4}dx \right]$. That is,

$$S \approx \frac{2S_{10} + \int_{11}^{\infty} \frac{1}{x^4}dx + \int_{10}^{\infty} \frac{1}{x^4}dx}{2} \approx 1.082328469$$

9.4 p-series

Any series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a fixed positive number.

Theorem 11. If $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p -series, then it converges if and only if $p > 1$.

Proof. We first consider the special case that $p = 1$. This series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *Harmonic series*. By the integral test $\int_1^{\infty} \frac{1}{x}dx \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x}dx = \lim_{t \rightarrow \infty} \ln x|_1^t = \infty$. This diverges and hence, the harmonic series diverges.

Now, consider the case $p \neq 1$:

$$\int_1^{\infty} \frac{1}{x^p}dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p}dx = \lim_{t \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^t = \frac{1}{1-p} \lim_{t \rightarrow \infty} \frac{1}{t^{p-1}} - 1$$

which diverges if $p - 1 < 0$ ($p < 1$) and converges if $p - 1 > 0$ ($p > 1$). □

9.5 Comparison test

Theorem 12 (Comparison test). *Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ be two series with $0 \leq a_n \leq b_n$. Then*

1. $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges
2. $\sum_{n=1}^{\infty} b_n$ diverges if $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Let A_n and B_n represent the n^{th} partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Suppose that $B_n \rightarrow L$. Then since $0 \leq a_n \leq b_n$ we have $0 \leq A_n \leq B_n$ for all $n > 0$. Now, A_n and B_n are both monotone increasing sequences and with $B_n \rightarrow L$ we have A_n is bounded above by L . By monotone sequence theorem, A_n converges.

For the second part, A_n is monotone increasing to infinity. Thus, also $B_n \geq A_n$ is also increasing to infinity. $\square$

9.6 Limit comparison test

Theorem 13 (Limit comparison test). *Suppose there are two series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ with both $a_n, b_n > 0$ for all n . Then*

1. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \neq 0, \text{ or } \infty$$

then either both converge or both diverge.

2. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$$

and $\sum b_n$ converges, then so does $\sum a_n$.

3. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$$

and $\sum b_n$ diverges, then so does $\sum a_n$.

Corollary 1. *If $\sum_{n=1}^{\infty} |a_n|$ converges then so does $\sum_{n=1}^{\infty} a_n$.*

Proof. Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges to M . We have the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

By the comparison test, then

$$\sum_{n=1}^{\infty} (a_n + |a_n|)$$

converges (since $\sum 2|a_n| = 2\sum |a_n| = 2M$). Assuming that $\sum (a_n + |a_n|) = L$ then $\sum a_n = L - M$ as required. $\square$

Definition 4. *If $\sum a_n$ is a series such that $\sum |a_n|$ converges, we say that $\sum a_n$ is absolutely convergent.*

If $\sum a_n$ converges, but $\sum |a_n|$ does not, then $\sum a_n$ is said to be conditionally convergent.

Any series of the form $\sum_n (-1)^n c_n$ where c_n 's are positive, then the series is called an alternating series.

Theorem 14 (Alternating series test). *Let $\sum_{n=1}^{\infty} (-1)^{n-1} c_n$ be an alternating series. If $c_{n+1} \leq c_n$ for all $n > 0$ and $\lim_{n \rightarrow \infty} c_n = 0$ then the series converges.*

Proof. Supposing that $\lim_{n \rightarrow \infty} c_n = 0$ and $0 \leq c_{n+1} \leq c_n$, let S_n denote the partial sums of the series $\sum_n (-1)^n c_n$. Note that S_{2n} is monotone increasing and bounded above by c_1 . So it remains to show that the limit of S_{2n} is the same as the sum of the series. Note that $S_{2n+1} = S_{2n} + c_{2n+1} \rightarrow S + 0 = S$ so thus, the series converges. $\square$

9.7 Alternating series estimation

Theorem 15 (Alternating series estimation theorem). *If $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series which satisfies*

$$0 \leq b_{n+1} \leq b_n \text{ and } \lim_{n \rightarrow \infty} b_n = 0$$

then,

$$R_n \leq |S - S_n| \leq b_{n+1}.$$

Proof. We know from above that S lies between two consecutive partial sums S_n and S_{n+1} . It follows that $|S - S_n| \leq |S_{n+1} - S_n| = b_{n+1}$. $\square$

Note; the rule that the error is smaller than the first neglected term is valid only for alternating series.

9.8 Ratio test

Let $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ for a series $\sum_n a_n$. Then

- if $\rho < 1$, the series is absolutely convergent and hence convergent
- if $\rho > 1$ or $= \infty$ then the series is divergent
- if $\rho = 1$ or does not exist then the test is inconclusive.

9.9 Root test

Let $\sum_n a_n$ be a series. Then

- if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ then $\sum a_n$ is absolutely convergent,
- if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$ the series is divergent
- and otherwise, if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ the test is inconclusive

10 Series testing strategies

The following is a recipe for testing convergence of series.

1. Check if $\sum a_n$ is geometric, harmonic, p-series or telescoping
2. Is the $\lim_{n \rightarrow \infty} a_n = 0$? If not, then the series diverges (by divergence test). Otherwise, convergence is still unknown
3. Does $|a_k|$ resemble r^k or $\frac{1}{k^p}$ for large k ? If so, try the comparison test, or limit comparison test with geometric series or a p-series
4. If $a_k = f(k)$ for some positive, decreasing continuous function f , try the integral test
5. If there are powers of n or $n!$ the ratio test is probably a good idea.
 - This test may result in the series diverges with $\rho > 1$ or $= \infty$ so there would be no need to proceed with alternating series.
 - This may show that $\sum_n a_n$ converges absolutely (when $\rho < 1$) hence converges
 - if $\rho = 1$ or doesn't exist, this test has failed and says nothing
6. Try the Root test, especially if you see powers of n !
 - If $\rho > 1$ or $= \infty$ series diverges
 - if $\rho < 1$ it converges absolutely and hence converges
 - if $\rho = 1$, or does not exist the try something else because we still don't know

7. if both ratio and root test have already failed, or it is clear that $\sum |a_n|$ is divergent, try (if series is alternating) the alternating series test in case it is conditionally convergent.

8. Error bounds:

- if series converges by the integral test, use the integral formula for approximate value of series.
- if the series converges by the alternating series test, use the alternating series estimation theorem to approximate the value.

11 Power series

A *power series* centered at a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ with x a variable. This series differs in values because of the variable x . This means that certain values cause series to converge and some won't. This should give the impression that a power series is a function whose domain is a collection of values x for which the series converges. When studying these, we will be interested in finding an elementary function f such that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ as well as determining its domain.

Observations: We already know that $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots$ converges to $\frac{1}{1-r}$ for $|r| < 1$ while it diverges everywhere else. So, we could say that $\sum_{n=0}^{\infty} x^n$, centered at zero converges to the function $\frac{1}{1-x}$ on the interval $(-1, 1)$ and diverges elsewhere.

The function $T_n(x) := \sum_{k=0}^n c_k(x-a)^k$ is a polynomial. Recall, from Taylor polynomials that if $T_1(x) = 1$, $T_2(x) = 1 + x$, $\dots$, $T_k(x) = 1 + x + x^2 + \dots + x^k$ that $\lim_{n \rightarrow \infty} T_n(x) = \frac{1}{1-x}$. Thus, a power series can be viewed as a function whose curve is a limit of a sequence of polynomials.

Also notice that a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges for $x = a$.

Here, we will mainly discuss two things;

1. For what values of x does a power series converge? (this is called the *radius and interval of convergence*)
2. To what function, if any, does it converge to?

Theorem 16. *The power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ converges either for $x = a$, for all x or there is a positive number R such that the series converges for all $|x-a| < R$ and diverges for all $|x-a| > R$.*

Note: When a power series is known to converge for all $|x-a| < R$, one must manually determine convergence at the endpoints of this open interval.

Proof. Given the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$, suppose we define $\rho = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$. Applying the ratio test to $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = \rho|x-a|$. Then this converges for $\rho|x-a| < 1$ or, equivalently, $|x-a| < 1/\rho$. So to say that $R = 1/\rho$. $\square$

Remark 2. *The above theorem states convergence on intervals centered at a . This is called the interval of convergence. Series may or may not converge at the endpoints of this interval. The $R = 1/\rho = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$. If the power series converges to a function, we say $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ for all $|x-a| < R$*

Example 38. *Discuss the convergence of the following power series*

1.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n} (x-2)^n$$

Solution: Here

$$R = 1/\rho = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n4^n} \frac{(n+1)4^{n+1}}{(-1)^{n+1}} \right| = \lim_{n \rightarrow \infty} |4 + 4/n| = 4$$

so $|x - 2| < 4$ or $-2 < x < 6$. At the endpoints; when $x = -2$ this is

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{n 4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

when $x = 6$ this becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is known to converge! Thus, the interval of convergence is $(-2, 6]$.

2.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n$$

Solution: Here we find $\rho = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} \right| = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0$ hence $1/\rho = \infty$ which implies radius of convergence is infinite and series converges for all values of x .

11.1 Functions as power series

Using that $\frac{1}{1-x} = \sum_{k \geq 0} x^k$ as a basis, we can find other functions expressed as power series!

For example,

1.

$$\frac{1}{3+x} = \frac{1}{3} \frac{1}{1 - (-\frac{x}{3})} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{-x}{3} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$$

and the interval of convergence is $\left| \frac{x}{3} \right| < 1$ or, $|x| < 3$.

2.

$$\frac{1}{1+x^2} = \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

with interval of convergence $x^2 < 1$ which is really just $|x| < 1$.

3.

$$\frac{x^5}{3+x^2} = \frac{x^5}{3} \frac{1}{1 - (-\frac{x^2}{3})} = \frac{x^5}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{3^{n+1}}$$

with radius of convergence $\left| \frac{x^2}{3} \right| < 1$ or $|x| < \sqrt{3}$

Now that we can deal with everything of this form, we will need a new style.

Theorem 17. Suppose $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$ and suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$. Then $f(x)$ is differentiable of $|x-a| < R$ and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int_a^x f(x) dx = \int_a^x \sum_{n=0}^{\infty} c_n (x-a)^n dx = \sum_{n=0}^{\infty} \int_a^x c_n (x-a)^n dx = \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1}$$

and both of these resulting power series have R for their radius of convergence.

Example 39. Show that $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$ with $R = 1$.

Solution:

$$\begin{aligned}\frac{d}{dx}(1-x)^{-1} &= \frac{1}{(1-x)^2} \\ &= \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=1}^{\infty} nx^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1)x^n\end{aligned}$$

$$\begin{aligned}\ln(1+x) &= \int_a^x \frac{1}{1+t} dt \\ &= \int_a^x \frac{1}{1-(-t)} dt \\ &= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n dt \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}\end{aligned}$$

and radius of convergence $R = 1$.

$$\begin{aligned}\arctan x &= \int_0^x \frac{1}{1+x^2} dx \\ &= \int_0^x \frac{1}{1-(-x^2)} dx \\ &= \int_0^x \sum_{n=0}^{\infty} (-1)^n x^{2n} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\end{aligned}$$

for $|x| < 1$.

11.2 Taylor and Mclaurin series

Suppose $f(x)$ has power series representation $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. What do the c_n 's look like?

Theorem 18. Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on $|x-a| < R$, then

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

That is to say

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

Example 40. We know that $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ which implies that

$$\frac{(-1)^n}{2n+1} = \frac{\arctan^{(n)}(0)}{n!}$$

so the n^{th} derivative of $\arctan$ evaluated at zero is $\frac{(-1)^n n!}{2n+1}$.

Definition 5 (Taylor series). Let $f(x)$ be a function which is differentiable infinitely many times, then the series

$$T_{f,a}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

is called the Taylor series generated by f centred at a .

Any Taylor series centred at 0, is called a McLaurin series.

Example 41. 1. $\sum_{n=0}^{\infty} x^n$ is the McLaurin series for $\frac{1}{1-x}$.

2. Find the McLaurin series generated by $f(x) = e^x$ and determine the interval of convergence.

Solution: We need only determine $c_n = \frac{f^{(n)}(0)}{n!} = \frac{e^x}{n!} \Big|_{x=0} = \frac{1}{n!}$. So

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Notice that there exists functions which generate taylor and McLaurin series that do not converge to the original function. For example, the piecewise defined function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

has a McLaurin series of 0, but this function is not the zero function.

The n^{th} degree taylor polynomial for f , centred at a is defined as the truncated version of its Taylor series $T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$. These are the partial sums of the Taylor series and again we can define the n^{th} remainder $R_n(x) = T_{f,a}(x) - T_n(x)$.

We have that

$$R_n(x) \rightarrow 0 \Leftrightarrow T_n(x) \rightarrow f(x)$$

To show that $R_n(x) \rightarrow 0$, we will use a tool called *Taylor's inequality*.

Theorem 19. Let $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ be a Taylor series with radius of convergence R . Then $|R_n(x)|$ can be bounded above in the following way:

If $|f^{(n+1)}(x)| \leq M_n$ on $|x-a| < R$, then

$$|R_n(x)| \leq \frac{M_n |x-a|^{n+1}}{(n+1)!}, \text{ on } |x-a| \leq d < R.$$

If we can show that $\lim_{n \rightarrow \infty} \frac{M_n |x-a|^{n+1}}{(n+1)!} = 0$ for all $|x-a| \leq d$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Example 42. Recall, McLaurin series of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has infinite radius of convergence (since $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges by divergence test).

Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x .

Solution: Consider any interval $[-k, k]$. Since e^x is increasing let $M_n = e^k$. Then Taylors inequality guarantees that $|R_n(x)| \leq \frac{e^k |x|^{n+1}}{(n+1)!}$. Now,

$$\lim_{n \rightarrow \infty} |R_n(x)| \leq e^k \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

as required.

Example 43. Find the McLaurin series generated by $\sin x$.

Solution: since the derivatives of sine are $\cos, -\sin, -\cos, \sin, \dots$ and repeat every four, we find the values of these derivatives to be $0, 1, 0, -1, \dots$ and repeat every four. Hence, then the series is given by

$$0 + 1 \cdot x + 0 \frac{x^2}{2!} - 1 \frac{x^3}{3!} + 0 \frac{x^4}{4!} + \dots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

and

$$R = \lim_{n \rightarrow \infty} \left| \frac{(2n+3)!}{(2n+1)!} \right| = \lim_{n \rightarrow \infty} (2n+3)(2n+2) = \infty$$

Does the series generated by $\sin x$ converge to $\sin x$?

We have

$$\sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!} (x-0)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and $|\sin^{(n)}(x)| \leq 1 = M_n$ for all $x \in [-k, k]$. Thus,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0$$

So the Taylor series generated by $\sin x$ converges to $\sin x$ for all x .

Note that if you wanted to approximate $\sin x$ near $x = 50$, choose $a = 50$.

Example 44. Estimate $\sin(0.1)$ with error $< 10^{-8}$.

Solution: Recall $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and we know that $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$. That is,

$$\begin{aligned} |\sin x - T_n(x)| &\leq \frac{|x|^{n+1}}{(n+1)!} < 10^{-8} \\ \frac{|0.1|^{n+1}}{(n+1)!} &< 10^{-8} \end{aligned}$$

Try $n = 6$, and find that $R_6(0.1) \leq \frac{(0.1)^7}{(6+1)!} = 1.9 \times 10^{-11} < 10^{-8}$. Thus, $T_6(0.1) = T_5(0.1) = (0.1) - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} = 0.099833$.

Example 45. Find the McLaurin series for $\cos x$.

Solution:

$$\cos x = \frac{d}{dx} \sin x = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Theorem 20. Let $\sum_n a_n x^n, \sum_n b_n x^n$ be two McLaurin series with radii of convergence R_a and R_b respectively. Then

$$\sum_n c_n x^n = \left(\sum_n a_n x^n \right) \cdot \left(\sum_n b_n x^n \right)$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{i=0}^n a_i b_{n-i}$$

has interval of convergence $R = \min\{R_a, R_b\}$.

Example 46. Find the McLaurin series for $\frac{1}{x-1} \ln(1-x)$.

Solution:

$$\frac{1}{x-1} = \frac{-1}{1-x} = (-1)(1+x+x^2+\cdots)$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

which implies

$$\begin{aligned} \frac{1}{x-1} \ln(1-x) &= (1+x+x^2+\cdots)(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots) \\ &= 1(0) + (1(0) + (1)(1))x + \left(1(0) + 1(\frac{1}{2}) + 1(1)\right)x^2 + (1 + 1/2 + 1/3)x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (1 + 1/2 + 1/3 + \cdots + 1/n) x^n \end{aligned}$$

and $R = 1$.

Everyone should know the following list of Taylor expansions

$$\begin{aligned} e^x &= \sum_{n \geq 0} \frac{x^n}{n!} \\ \cos x &= \sum_{n \geq 0} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x &= \sum_{n \geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ \arctan x &= \sum_{n \geq 0} \frac{(-1)^n x^{2n+1}}{2n+1} \\ \frac{1}{1-x} &= \sum_{n \geq 0} x^n \\ \ln(1-x) &= - \sum_{n \geq 0} \frac{x^{n+1}}{n+1} \end{aligned}$$

12 Applications of Taylor and McLaurin series

1. Find the sum of the series $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots = \sum_{k \geq 0} \frac{(-1)^k \pi^{2k}}{(2k)!} = \cos(\pi) = -1$.
2. Approximate the integral $\int_0^1 \frac{1-\cos x}{x^2} dx$ with error less than 0.00001.

Solution:

$$\begin{aligned}
 \int_0^1 \frac{1-\cos x}{x^2} dx &= \int_0^1 \frac{1}{x^2} (1 - (1 - x^2/2! + x^4/4! - x^6/6! + \dots)) dx \\
 &= \int_0^1 \frac{1}{x^2} (x^2/2! - x^4/4! + x^6/6! + \dots) dx \\
 &= \int_0^1 (1/2! - x^2/4! + x^4/6! + \dots) dx \\
 &= \left[\frac{x}{2} - \frac{x^3}{3 \cdot 4!} + \frac{x^5}{5 \cdot 6!} + \dots \right]_0^1 \\
 &= \frac{1}{2!} - \frac{1}{3 \cdot 4!} + \frac{1}{5 \cdot 6!} - \frac{1}{7 \cdot 8!} + \dots
 \end{aligned}$$

This is an alternating series, so the error bound for S_n is c_{n+1} . Letting $n = 4$, then $c_4 = \frac{1}{7 \cdot 8!} = 0.0000036 < 0.00001$. Therefore S_3 is appropriate and

$$S_3 = \frac{1}{2!} - \frac{1}{3 \cdot 4!} + \frac{1}{5 \cdot 6!} \approx 0.486388$$

3. Approximate the integral $\int_0^1 \frac{1-\cos x}{x^2} dx$ with error less than 10^{-7} .

Solution:

$$\left| \sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \right| \leq |R_n(x)|$$

implies

$$\left| \frac{\sin x}{x} - \left(\frac{1}{x} \right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \right| \leq \left| \frac{R_n(x)}{x} \right|$$

so

$$\begin{aligned}
 \left| \int_0^{1/2} \frac{\sin x}{x} dx - \int_0^{1/2} \left(\frac{1}{x} \right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) dx \right| &\leq \int_0^{1/2} \left| \frac{R_n(x)}{x} \right| dx \\
 &\leq \int_0^{1/2} \frac{1}{x} \frac{x^{n+1}}{(n+1)!} dx \\
 &\leq \int_0^{1/2} \frac{x^n}{(n+1)!} dx \\
 &= \left[\frac{x^{n+1}}{(n+1)(n+1)!} \right]_0^{1/2} \\
 &= \frac{1}{2^{n+1}(n+1)(n+1)!}
 \end{aligned}$$

When $n = 6$, this is $\frac{1}{2^7(7)7!}$.

12.1 Binomial series

Recall, if $n > 0$ that

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Definition 6. For α and real number and k a positive integer, define $\binom{\alpha}{k}$ as follows:

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}$$

Theorem 21 (Binomial Series Theorem). Let $\alpha \in \mathbb{R}$. Then the McLaurin series for

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

with interval of convergence $(-1, 1)$.

Example 47. Find a McLaurin series for $\frac{1}{\sqrt{9-x}}$.

Solution:

$$\begin{aligned} \frac{1}{\sqrt{9-x}} &= \frac{1}{3} (1 + (-x/9))^{-1/2}, \quad \text{let } u = x/9 \\ &= \frac{1}{3} \left(1 + \frac{-1/2}{1!}(-u) + \frac{(-1/2)(-1/2-1)}{2!}(-u)^2 + \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!}(-u)^3 + \dots \right) \\ &= \frac{1}{3} \left(1 + \frac{1}{2}u + \frac{1 \cdot 3}{2^2 2!}u^2 + \frac{1 \cdot 3 \cdot 5}{2^3 3!}u^3 + \dots + \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n!}u^n + \dots \right) \\ &= \frac{1}{3} \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n!} u^n \end{aligned}$$

So, now the McLaurin series for $f(x) = \frac{1}{3} \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n! 9^n} x^n$ we can compute $f^{(12)}(0)$ since we know the series! Indeed,

$$\frac{f^{(12)}(0)}{12!} = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{3 \cdot 2^{12} 12! 9^n} \Rightarrow f^{(12)}(0) = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{3 \cdot 2^{12} \cdot 9^n}$$

12.2 One last application

The pendulum of length L ;

$$T = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} d\phi$$

We would like to approximate T given ϕ_0

Express

$$\frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n!} (k^2 \sin^2 \phi)^n = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n!} k^{2n} \sin^{2n} \phi$$

Then

$$4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} d\phi = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n!} k^{2n} \sin^{2n} \phi d\phi$$